Rare Shock, Two-Factor Stochastic Volatility and Currency Option Pricing

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ABSTRACT In this paper, we develop an option valuation model where the dynamics of the spot foreign exchange rate is governed by a two-factor Markov-modulated jump-diffusion process. The short-term fluctuation of stochastic volatility is driven by a Cox–Ingersoll–Ross (CIR) process and the long-term variation of stochastic volatility is driven by a continuous-time Markov chain which can be interpreted as economy states. Rare events are governed by a compound Poisson process with log-normal jump amplitude and stochastic jump intensity is modulated by a common continuous-time Markov chain. Since the market is incomplete under regime-switching assumptions, we determine a risk-neutral martingale measure via the Esscher transform and then give a pricing formula of currency options. Numerical results are presented for investigating the impact of the long-term volatility and the annual jump intensity on option prices.

KEY WORDS: Spot foreign exchange rate, rare shock, Esscher transform, currency option

1. Introduction

Since the termination of the Bretton Woods regime in 1971, major currencies have started to implement floating exchange rate system. For hedging the fluctuating risk, a variety of currency options have been designed and priced. For instance, Biger and Hull (1983) and Garman and Kohlhagen (1983) provide an explicit pricing formula of a European-style currency option under the assumption that the dynamics of the spot foreign exchange (FX) rate is modelled by a geometric Brownian motion. However, the Black–Scholes framework does not suffice to explain rare shocks and stochastic volatility, which emphasizes the need for more complicated option pricing models. Merton (1976) employs compound Poisson processes to model discontinuous trading. Shastri and Wetheyavivorn (1987) investigate the spot FX rate using jump-diffusion processes. Heston (1993) considers stochastic volatility governed by a
Cox–Ingersoll–Ross (CIR) process. Some other works include Melino and Turnbull (1990), Bates (1996) and so on.

Recently, Markov-modulated models attract more and more attention as financial dynamics, where market parameters are modulated by a hidden Markov chain. The hidden Markov model has emerged as a plausible model for some interesting applications in finance. It can be used to model the evolution of the states of an economy, and to model both interest rate and volatility. In practice, the state of a hidden Markov chain is not observable. Many previous works assume that the information about the whole path of the hidden Markov chain up to maturity T is known at any time. Markov-modulated (regime-switching) market parameters are adopted to investigate the time-inhomogeneity in the financial market (see, e.g., Elliott, Chan, & Siu, 2005; Elliott & Osakwe, 2006; Jobert & Rogers, 2006). Siu, Yang, and Lau (2008) developed a two-factor Markov-modulated stochastic volatility model. The Markov-modulated dividends models are studied by Di Graziano and Rogers (2009). Bo, Wang, & Yang (2010) considered a Markov-modulated stochastic volatility jump-diffusion model, with stochastic volatility component driven by a continuous-time finite-state Markov chain.

In this paper, we model the dynamics of the spot FX rate by a two-factor Markov-modulated stochastic volatility jump-diffusion process which combines short-term fluctuation, long-term fluctuation and rare events together. Volatility risk of the spot FX rate can be decomposed into long-term risk and short-term risk (see, e.g., Siu et al., 2008). The short-run fluctuation of stochastic volatility exhibits the characteristic of mean-reversion (see, e.g., Heston, 1993), while the long-run variation of stochastic volatility depends on the state of the economy (see, e.g., Siu et al., 2008). Rare shocks do exist in FX markets and have a significant impact on the spot FX rate. The jump-diffusion structure is appropriate to model the spot FX rate (see, e.g., Bates, 1996; Bo et al., 2010; Johnson & Schneewels, 1994). The combination of short-term risk, long-term risk and rare shocks is reasonable for both pricing option and describing stochastic volatility.

In our setting, the short-term fluctuation of stochastic volatility is driven by a CIR process (see, e.g., Cox, Ingersoll, & Ross, 1985; Heston, 1993). Rare events are described as a compound Poisson process with log-normal jump amplitude and stochastic jump intensity. The long-term variation of stochastic volatility and stochastic jump intensity are driven by a continuous-time hidden Markov chain which can be interpreted as the states of an economy. In general, one may interpret the state as a macroeconomic index, such as gross domestic product and different stages of business cycles.

It is worth noting that the market is incomplete under regime-switching assumptions, which leads to infinitely many equivalent martingale measures when pricing options. Therefore, a variety of studies have been conducted for determining an equivalent martingale measure. An equivalent martingale measure is identified in Davis (1997) by solving a utility maximization problem. Gerber and Shiu (1994) first employ the Esscher transform to price options when the dynamics of the underlying asset is governed by a class of stochastic processes with stationary and independent increments. This paper adopts the random Esscher transform technique (see, e.g., Elliott et al., 2005; Elliott, Siu, Chan, & Lau, 2007) by decomposing the log spot FX rate into a continuous diffusive part and a jump part, and then identifies their respective
risk premiums. Finally, we present some numerical simulations by Monte Carlo simulations to investigate how the long-term volatility and the annual jump intensity affect option prices.

The rest of this paper is organized as follows. In Section 2, a two-factor Markov-modulated jump-diffusion model for the FX rate is proposed. The existence of equivalent martingale measures for no-arbitrage pricing is shown in Section 3. In Section 4, we consider the valuation of European-style currency options. Numerical results are illustrated in Section 5. The concluding remarks are contained in Section 6.

2. The Dynamics of the Spot FX Rate

In this section, we propose the dynamics of the spot FX rate. It is often claimed that one of the products of the current system of floating exchange rates is exchange rate volatility. Exchange rate volatility refers to short-run fluctuations of the exchange rate around its mean value. The shift to floating exchange rates has indeed produced significant nominal exchange rate volatility. While, the exchange rate misalignment refers to the deviation of the real exchange rate from its long-run equilibrium value (see, e.g., Moosa, 2005), which may result in long-term risk. Some works consider the effect of mean-reversion property of the spot FX rate and the volatility on option prices (see, e.g., Wong & Lo, 2009; Wong & Zhao, 2010). This paper mainly focuses on jump risk and stochastic volatility under regime-switching conditions through comparing with the results of Bo et al. (2010) and Siu et al. (2008), in which the dynamics of the spot FX rate are expanded upon geometric Brownian motion framework. Hence, we do not incorporate the mean-reversion factor into the dynamics of the spot FX rate. To explain our model intuitively, we list an example first. Figure 1 depicts the daily data for US dollar/Japanese yen (USD/JPY) spot FX rate, the exchange rate of the USD against the JPY. From the left panel, jump risk can be obviously observed during 1984. Without jump risk, short-term volatility is overestimated in Siu et al. (2008). Turn to Markov-modulated stochastic volatility jump-diffusion model in Bo et al. (2010), where stochastic volatility is modelled by a CIR process. From the right panel in Figure 1, the 20-day’s short-term volatility has a mean-reverting path. By calculation, the long-term volatility during 1982–1989 is 50.8, while it dramatically decreases to

![Figure 1](image-url)
Consider a continuous-time financial model with two different currencies, the domestic currency and the foreign currency. Suppose that the domestic and the foreign instantaneous interest rates are linked to the current economic environment. That is to say, the value of instantaneous interest rates varies with economic states. To model the states of the economy, we adopt a continuous-time, finite-state Markov chain $\xi = (\xi_t)_{0 \leq t \leq T}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a real-world probability measure. In particular, we could assume there are just two states for $\xi$, which represent ‘bull’ and ‘bear’. Instantaneous interest rates have high values when the economy is ‘bull’, and have low values when the economy is ‘bear’. Without loss of generality, we take the state space of $\xi$ to be a finite set of unit vectors $(e_1, e_2, \ldots, e_n)$ with $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^n$ (see, e.g., Elliott, Aggoun, & Moore, 1994). The domestic instantaneous interest rates $(r^D_t)_{0 \leq t \leq T}$ and the foreign instantaneous interest rates $(r^F_t)_{0 \leq t \leq T}$ have the following forms:

$$
\begin{align*}
  r^D_t &= \langle r^D, \xi_t \rangle, \\
  r^F_t &= \langle r^F, \xi_t \rangle,
\end{align*}
$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^n$. Intuitively, the domestic instantaneous interest rate takes the value $r^D$ when the economy stays in state $i$.

In this paper, the dynamics of the spot FX rate is assumed to follow a two-factor Markov-modulated jump-diffusion process with stochastic volatility, which contains rare shocks, short-term fluctuation and long-term fluctuation. Precisely, the dynamics of the spot FX rate could be expressed mathematically as follows.

Let $W^1$ and $W^2$ be two independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. Denote $N = (N_t)_{0 \leq t \leq T}$ as a Poisson process with Markov-modulated intensity $(\lambda_t)_{0 \leq t \leq T}$, i.e.,

$$
\lambda_t = \langle \lambda, \xi_t \rangle, \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in (0, \infty)^n.
$$

We further assume that $W^1$, $W^2$, $N$ and $\xi$ are mutually independent. Then, the spot FX rate $S_t$ is governed by the following stochastic differential equation:

$$
\begin{align*}
  \frac{dS_t}{S_t^-} &= (\alpha_t - k\lambda_t)dt + \sqrt{V_t}dW^1_t + \sigma_t dW^2_t + (e^{Z_t} - 1)dN_t, \\
  dV_t &= (\gamma - \beta V_t)dt + \sigma_t \sqrt{V_t}dW^1_t, \\
  \text{Cov}(dW^1_t, dW^1_t) &= \rho dt,
\end{align*}
\tag{2.1}
$$

where the appreciation rate $(\alpha_t)_{0 \leq t \leq T}$ and the long-term part of stochastic volatility $(\sigma_t)_{0 \leq t \leq T}$ are assumed to be modulated by a common Markov chain $\xi = (\xi_t)_{0 \leq t \leq T}$ (see, e.g., Elliott & Osakwe, 2006),

$$
\begin{align*}
  \alpha_t &= \langle \alpha, \xi_t \rangle, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n, \\
  \sigma_t &= \langle \sigma, \xi_t \rangle, \quad \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in (0, \infty)^n.
\end{align*}
$$
For the jump arrival process $N = (N_t)_{0 \leq t \leq T}$, the jump amplitude is controlled by $Z_t$, which is normally distributed with mean $\mu_J$ and standard deviation $\sigma_J$. Given the jump arrival, the mean percentage jump of the spot FX rate is $k = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$. Specifically, we assume that for $t \neq s$, $Z_t$ and $Z_s$ are independent. The short-term variance of stochastic volatility $V_t$ is modelled by a mean-reverting square root process. Further, $W, W^1, W^2, N$ and $\xi$ are assumed to be independent of $Z = (Z_t)_{0 \leq t \leq T}$.

The spot FX rate process converts foreign cash flows into units of the domestic currency. Denote $S^d_t$ as the discounted spot FX rate at time $t$ by the current value of the domestic money market account. Thus, $S^d_t$ can be represented as

$$S^d_t = \exp\left(\int_0^t (r^F_s - r^D_s) ds\right) S_t, \quad 0 \leq t \leq T.$$

Further, Itô rules for general semi-martingales (see, e.g., Protter, 1990) imply that $S^d_t$ satisfies the following stochastic differential equation:

$$\frac{dS^d_t}{S^d_t} = (r^F_t - r^D_t + \alpha_t - k\lambda_t) dt + \sqrt{V_t} dW^1_t + \sigma_t dW^2_t + (e^{Z_t} - 1) dN_t. \quad (2.2)$$

Based on the expression of $S^d_t$, we will select an equivalent martingale measure and then price European currency options.

### 3. Equivalent Domestic Martingale Measures

Since the market described in Section 2 is incomplete, our aim is to determine an equivalent martingale measure through the random Esscher transform (see, e.g., Bo et al., 2010; Elliott et al., 2005, 2007). We decompose the log spot FX rate $Y_t = \log S^d_t$ $(0 \leq t \leq T)$ into a continuous part and a jump part and identify their respective risk premiums. According to Itô rules for general semi-martingales,

$$Y_t = C_t + J_t, \quad (3.1)$$

where the continuous part $C_t$ and the jump part $J_t$ admit the following forms:

$$C_t = \int_0^t \left(-\frac{1}{2} V_s + \alpha_s - k\lambda_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sqrt{V_s} dW^1_s + \int_0^t \sigma_s dW^2_s,$n

$$J_t = \int_0^t Z_s dN_s.$$

Let $(\mathcal{F}_t^\xi)_{0 \leq t \leq T}$ and $(\mathcal{F}_t^Y)_{0 \leq t \leq T}$ be the $\sigma$-algebras generated by $\xi$ and $Y$. The current price of currency options depends on the current state of $\xi$, which is unobservable. Then, we need to price options under a new measure, which should contain the information of the whole path of the Markov chain. In addition, the information of short-term stochastic volatility till maturity time $T$ is also needed as in Siu et al. (2008) and Bo et al. (2010). Hence, we denote $\mathcal{G}_t = \mathcal{F}_t^\xi \vee \mathcal{F}_t^Y$ and $\mathcal{H}_t = \mathcal{F}_t^Y \vee \mathcal{G}_T$ for
0 \leq t \leq T$, and then define an Esscher transform $Q^{θ^c,θ^d} \sim P$ on $\mathcal{H}_t$ with respect to two families of regime-switching parameters $(θ^c_0)_{0 ≤ s ≤ t}$ and $(θ^d_0)_{0 ≤ s ≤ t}$ by

$$L_t^{θ^c,θ^d} = \frac{dQ^{θ^c,θ^d}}{dP} \bigg|_{\mathcal{H}_t} = \exp \left( \int_0^t θ^c_s dC_s + \int_0^t θ^d_s dJ_s \right)
\frac{1}{\mathbb{E} \left[ \exp \left( \int_0^t θ^c_s dC_s + \int_0^t θ^d_s dJ_s \right) \bigg| \mathcal{G}_t \right]}.$$

(3.2)

where $θ^m_t = (θ^m_0, ξ_t)$, and $θ^m = (θ^m_1, θ^m_2, ..., θ^m_n) \in \mathbb{R}^n$ for $m \in \{c, J\}$. Substituting the expressions of $C_t$ and $J_t$ into Equation (3.2), we can get a specific form of $(L_t^{θ^c,θ^d})_{0 ≤ t ≤ T}$.

**Proposition 3.1.** For $0 \leq t \leq T$, the Radon–Nikodym derivative of the regime switching Esscher transform $L_t^{θ^c,θ^d}$ can be written as

$$L_t^{θ^c,θ^d} = \exp \left( \int_0^t θ^c_s \sqrt{V_s} dW^1_s + \int_0^t θ^d_s \sigma_s dW^2_s - \frac{1}{2} \int_0^t (V_s + σ^2_s)(θ^c_s)^2 ds \right)
× \exp \left( \int_0^t θ^d_s Z_s dN_s - \int_0^t s (e^{θ^d_s (1 + θ^c_s)^2} - 1) ds \right).$$

To derive the martingale condition, we assume that $\mathbb{E}e^{\frac{1}{2} \int_0^T (θ^c_s)^2 V_s ds} < ∞$ and $\mathbb{E}e^{\frac{1}{2} \int_0^T (1 + θ^c_s)^2 V_s ds} < ∞$, which in turn guarantee that $(L_t^{θ^c,θ^d})_{0 ≤ t ≤ T}$ is an exponential $(\mathcal{H}_t)_{0 ≤ t ≤ T}$-martingale. Under these assumptions, the following lemma gives a necessary and sufficient condition such that the martingale condition is satisfied.

**Lemma 3.1.** The martingale condition holds if and only if the Markov-modulated parameters $(θ^c_0, θ^d_0)_{0 ≤ t ≤ T}$ satisfy, for all $0 ≤ t ≤ T$,

$$r^F_t - r^D_t - α_t - kλ_t + θ^c_t(V_t + σ^2_t) + θ^d_t k_t = 0,$$

(3.3)

where the random Esscher transform intensity $λ^θ_t$ of the Poisson process and the mean percentage jump size $k^θ_t$ are, respectively, given by

$$λ^θ_t = λ_t e^{θ^d_t μ_t + \frac{1}{2} (θ^d_t σ_t)^2}, \quad k^θ_t = (k + 1) e^{θ^d_t σ_t} - 1.$$

In the following, we give an explicit pair of solutions $(θ^c_0, θ^d_0)_{0 ≤ t ≤ T}$, which satisfy the martingale condition (3.3) and the assumptions $\mathbb{E}e^{\frac{1}{2} \int_0^T (θ^c_s)^2 V_s ds} < ∞$ and $\mathbb{E}e^{\frac{1}{2} \int_0^T (1 + θ^c_s)^2 V_s ds} < ∞$, and identify the spot FX rates dynamics $S_t$ under the equivalent domestic martingale measure $Q^{θ^c,θ^d}$.

**Proposition 3.2.** Define

$$θ^c_t = \frac{kλ_t - α_t + r^D_t - r^F_t}{V_t + σ^2_t},$$

(3.4)

$$θ^d_t = -\frac{μ_t + \frac{1}{2} σ^2_t}{σ^2_t}, \quad 0 ≤ t ≤ T.$$

(3.5)
Then \((\theta_{t,x}^{c}, \theta_{t,j}^{s})_{0 \leq t \leq T}\) satisfy the martingale condition (3.3), \(\mathbb{E}e^{\int_0^T (\theta_{s,x}^{c}-\theta_{s,j}^{s})v_s ds} < \infty\) and \(\mathbb{E}e^{\int_0^T (1+\theta_{s,j}^{s})v_s ds} < \infty\). Moreover, the random Esscher transform mean percentage jump size and the intensity of the Poisson process are, respectively, given by

\[
\kappa_t^{\theta,x,j} = (k + 1)e^{\theta_t^{c,x}\sigma_j^2 - 1}. \tag{3.6}
\]

\[
\lambda_t^{\theta,x,j} = \lambda_t \exp \left( -\frac{\mu_j^2}{2\sigma_j^2} + \frac{\sigma_j^2}{8} \right). \tag{3.7}
\]

Actually, there are infinitely many solutions to the martingale condition (3.3). Here, we determine the parameters in Equations (3.4) and (3.5) as a specific solution. Similar to Elliott et al. (2007) and Bo et al. (2010), \((\theta_{t,x}^{c}, \theta_{t,j}^{s})_{0 \leq t \leq T}\) are chosen to make the random Esscher transform mean percentage jump size zero. Finally, Girsanov theorem helps us to derive the dynamics of the spot FX rates under \(Q^{\theta^{c,x}, \theta^{s,j}}\).

**Lemma 3.2.** Conditional on \(G_T\), it holds that

\[
\tilde{W}_t^{1,\theta^{c,x}} = W_t^1 - \int_0^t \sqrt{V_s} \theta_{s,x}^{c} ds, \quad 0 \leq t \leq T, \tag{3.8}
\]

\[
\tilde{W}_t^{2,\theta^{s,j}} = W_t^2 - \int_0^t \theta_{s,j}^{s} \sigma_s ds, \quad 0 \leq t \leq T \tag{3.9}
\]

are Brownian motions under \(Q^{\theta^{c,x}, \theta^{s,j}}\). The intensity \(\lambda_t^{\theta,x,j}\) and mean percentage jump size \(k_t^{\theta,x,j}\) of the Poisson process \(N\) admit,

\[
\lambda_t^{\theta,x,j} = \lambda_t e^{\theta_t^{c,x}\mu_j + \frac{1}{2}(\theta_t^{c,x})^2}, \tag{3.10}
\]

\[
k_t^{\theta,x,j} = (k + 1)e^{\theta_t^{c,x}\sigma_j^2 - 1}. \tag{3.11}
\]

Now the dynamics of the spot FX rate \(S\) admits the following form, under \(Q^{\theta^{c,x}, \theta^{s,j}}\):

\[
\begin{align*}
\frac{dS_t}{S_t} &= (\alpha_t^{\theta,x} - k \lambda_t^{\theta,x,j} \lambda_t^{\theta,x,j})dt + \sqrt{V_t} d\tilde{W}_t^{1,\theta^{c,x}} + \sigma_t d\tilde{W}_t^{2,\theta^{s,j}} + (\varepsilon_{Z_t} - 1)dN_t, \\
\frac{dV_t}{V_t} &= (\gamma - \beta_t^{\theta,x} V_t)dt + \sigma_{\gamma} \sqrt{V_t} d\tilde{W}_t^{\theta^{c,x}}, \\
\tilde{W}_t^{\theta^{c,x}} &= \rho \tilde{W}_t^{1,\theta^{c,x}} + \sqrt{1 - \rho^2} W_t, \quad \rho_{\gamma} = \rho_{\sigma_{\gamma}}
\end{align*} \tag{3.12}
\]

where \(\alpha_t^{\theta,x} = \alpha_t - k \lambda_t + \theta_{t,x}(V_t + \sigma_t^2)\) and \(\beta_t^{\theta,x} = \beta - \rho_{\sigma_t} \theta_{t,x}^{c} \) for \(0 \leq t \leq T\).

Up to now, we have determined an equivalent domestic martingale measure \(Q^{\theta^{c,x}, \theta^{s,j}}\) and got the dynamics of the spot FX rate \(S\) under \(Q^{\theta^{c,x}, \theta^{s,j}}\). In the following sections, we will adopt Equation (3.12) to price European currency options and perform numerical simulations to investigate how the long-term volatility \(\sigma\) and the annual jump intensity \(\lambda\) affect option prices.
4. Valuation of European Currency Options

In this section, we investigate the valuation of European currency options with strike price $K$ and maturity $T$. We provide an integral representation for option prices under the assumption that $\rho = 0$.

Let us return to Equation (3.12). Recall that $\mathcal{F}_T^\xi$ and $\mathcal{F}_T^V$ represent the $\sigma$-algebras generated by the processes $\xi$ and $V$ up to and including time $t$. Conditional on $\mathcal{F}_T^\xi \vee \mathcal{F}_T^V$, define a standard Brownian motion $B_t^\xi \triangleq \frac{\sqrt{\mathcal{V}_t \sigma^2}}{\sqrt{\mathcal{V}_t + \sigma^2}} \tilde{W}_t^{1,\theta^*} + \frac{\sigma_t}{\sqrt{\mathcal{V}_t + \sigma^2}} \tilde{W}_t^{2,\theta^*}$, which is independent of $N$ and $\xi$. Due to the independence between $\tilde{W}_t^{1,\theta^*}$ and $\tilde{W}_t^{2,\theta^*}$, we can rewrite the dynamics of the spot FX rate as follows:

$$
\frac{dS_t}{S_t} = (\alpha^\theta_t - k^\theta_t J_t^\theta - \lambda^\theta_t J_t^\theta) dt + \sqrt{\theta_t} dW_t + (e^{Z_{t-}} - 1)dN_t. \tag{4.1}
$$

Note that given $\mathcal{F}_T^\xi \vee \mathcal{F}_T^V$, the risk-neutral spot FX rate in Equation (4.1) is consistent with the discontinuous stock return model in Merton (1976). For the European call currency options with strike price $K$ and maturity $T$, the conditional price given $\mathcal{F}_T^\xi \vee \mathcal{F}_T^V$ at time zero admits

$$
C_0(S_0, V_0, K, T; V, \xi) = \mathbb{E}^{\mathbb{Q}} e^{-\int_0^T (r^D_s - r^F_s)ds} (S_T - K)^+ | \mathcal{F}_T^\xi \vee \mathcal{F}_T^V, \tag{4.2}
$$

where $S_0$ and $V_0$ are initial values of $S_t$ and $V_t$, respectively. Let $J_i(t, T)$ denote the occupation time of $\xi$ in state $i (i = 1, 2, \ldots, n)$ over the time period $[t, T]$ with $t < T$. Then, we have

$$
R_{i,T} = \frac{1}{T-t} \int_t^T (r^D_s - r^F_s)ds = \frac{1}{T-t} \sum_{i=1}^n (r^D_t - r^F_t) J_i(t, T);
$$

$$
U_{i,T} = \frac{1}{T-t} \int_t^T \sigma^2_s ds = \frac{1}{T-t} \sum_{i=1}^n \sigma^2_s J_i(t, T);
$$

$$
\lambda^\theta_{i,T} = \frac{1}{T-t} \int_t^T \lambda^\theta_s J_s ds = \frac{1}{T-t} \sum_{i=1}^n \lambda^\theta_{i} J_i(t, T);
$$

$$
\lambda^\theta_{i,T} = \frac{1}{T-t} \int_t^T (1 + k^\theta_{i}) \lambda^\theta_s J_s ds = \frac{1}{T-t} \sum_{i=1}^n (1 + k^\theta_{i}) \lambda^\theta_{i} J_i(t, T);
$$

$$
V^2_{i,T,m} = U_{i,T} + \frac{m \sigma^2_j}{T-t};
$$
\[ R_{t,T,m} = R_{t,T} - \frac{1}{T-t} \int_t^T \lambda_s^\text{opt} J_s^\text{opt} \, ds + \frac{m}{T-t} \int_t^T \frac{\log(1+k_s^\text{opt} J_s)}{T-t} \, ds \]

\[ = R_{t,T} - \frac{1}{T-t} \sum_{i=1}^n \lambda_i^\text{opt} J_i^\text{opt} J(t, T) + \frac{m}{T-t} \sum_{i=1}^n \frac{\log(1+k_i^\text{opt} J_i)}{T-t} J_i(t, T); \]

\[ I_{t,T} = \frac{1}{T-t} \int_t^T V_s \, ds, \]

where \( m \) denotes the number of jumps in the time interval \([t, T]\). Apply the pricing formula in Merton (1976) and define

\[ \tilde{C}(S_0, V_0, K, T; R_{0,T}, U_{0,T}, \lambda_{0,T}^\text{opt}, I_{0,T}) \]

\[ = \sum_{m=0}^\infty \frac{e^{-T\lambda_0^2}}{m!} BS_0(S_0, K, T, V_0^2 + I_{0,T}, R_{0,T,m}), \]

where \( BS_0(S, K, T, \sigma^2, r) \) denotes the standard Black–Scholes option pricing formula with initial spot FX rate \( S_0 \), strike price \( K \), risk-free rate \( r \), volatility square \( \sigma^2 \) and maturity \( T \). Denote \( p(v) \) as the conditional distribution of \( I_{0,T} \) given the initial value \( V_0 \), then we have

\[ C(S_0, V_0, K, T; R_{0,T}, U_{0,T}, \lambda_{0,T}^\text{opt}) = \int_0^\infty \tilde{C}(S_0, V_0, K, T; R_{0,T}, U_{0,T}, \lambda_{0,T}^\text{opt}, v) p(v) \, dv. \]

Therefore, we get the European call option pricing formula as follows:

\[ C(S_0, V_0, K, T) = \int_{[0,T]} C(S_0, V_0, K, T; R_{0,T}, U_{0,T}, \lambda_{0,T}^\text{opt}) \psi(J_1, J_2, \ldots, J_n) dJ_1 dJ_2 \ldots dJ_n, \]

where \( \psi(J_1, J_2, \ldots, J_n) \) denotes the joint probability distribution density for the occupation time \((J_1(0, T), J_2(0, T), \ldots, J_n(0, T))\). Moreover, the occupation time can be completely determined by the following characteristic function:

\[ E \left[ e^{i\langle J(0, T) \rangle} \bigg| \mathcal{F}_T^S \right] = \left( \xi_t e^{(Q+iD)T}, 1 \right), \quad \text{(4.3)} \]

where \( i = \sqrt{-1}, \) \( 1 = (1, 1, \ldots, 1) \in \mathbb{R}^n \) and the matrix \( D \) denotes a diagonal matrix consisting of the elements in the vector \( t = (t_1, t_2, \ldots, t_n) \) as its diagonal.

### 5. Numerical Simulations

In this section, we perform a numerical experiment by Monte Carlo simulations. We compare numerical results of the two-factor jump-diffusion model with those of
Bo et al. (2010) and Siu et al. (2008), and then investigate how the long-term volatility $\sigma_t$ and the annual jump intensity $\lambda_t$ affect option prices.

By Itô formula for general semi-martingales, the log spot FX rate $Y = (Y_t)_{0 \leq t \leq T}$ satisfies:

$$
\begin{align*}
\frac{dY_t}{Y_t} &= (r^D_t - r^F_t - \frac{1}{2} \sigma_t^2)dt + \sqrt{V_t}d\tilde{W}^{1,\theta_{t}}_t + \sigma_t d\tilde{W}^{2,\theta_{t}}_t + Z_t dN^J_t, \\
\frac{dV_t}{V_t} &= \gamma - \beta \sigma_t dt + \sigma_v \sqrt{V_t}d\tilde{W}^{\sigma}_{t}, \\
\frac{d\tilde{W}^{1,\theta_{t}}_t}{\tilde{W}^{1,\theta_{t}}_t} &= \rho \tilde{W}^{1,\theta_{t}}_t + \sqrt{1 - \rho^2} \tilde{W}^{\sigma}_{t}, \\
\frac{d\tilde{W}^{2,\theta_{t}}_t}{\tilde{W}^{2,\theta_{t}}_t} &= \rho \tilde{W}^{2,\theta_{t}}_t, \\
\frac{d\tilde{W}^{\sigma}_{t}}{\tilde{W}^{\sigma}_{t}} &= \rho \tilde{W}^{\sigma}_{t},
\end{align*}
$$

where $\tilde{W}^{1,\theta_{t}}_t = (\tilde{W}^{1,\theta_{t}}_0)_{0 \leq t \leq T}$ and $\tilde{W}^{2,\theta_{t}}_t = (\tilde{W}^{2,\theta_{t}}_0)_{0 \leq t \leq T}$ are two independent Brownian motions under $Q^{\theta_{t}}$, and the Poisson process $N^{J}_{0 \leq t \leq T}$ has the Esscher transform intensity $\lambda_{t}^{\theta_{t}} = \lambda_t \exp \left( -\frac{\mu^2 J}{2 \sigma^2 J} + \sigma^2 J/8 \right)$.  

In order to observe the impact of rare shocks, we compare our model with the two-factor Markov-modulated diffusion model in Siu et al. (2008). Meanwhile, comparing with Bo et al. (2010), we can find how the long-term fluctuation affects option prices. We quote the parameter values from Siu et al. (2008) and perform 10,000 simulations for computing option prices.

In the numerical simulations, we assume that the transition probability matrix of the two-state Markov chain $\xi$ is given by

$$
\begin{pmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{pmatrix} = \begin{pmatrix}
0.7 & 0.3 \\
0.2 & 0.8
\end{pmatrix}.
$$

Moreover, we denote two states by $e_1$ when the domestic economy is booming, and by $e_2$ when the economy encounters recession. The parameters values are listed in Tables 1 and 2.

We fix the maturity years $T = 0.5, 1.0$ and $1.5$, respectively. Then, we consider a range of spot-to-strike ratio $S_0/K$ from 0.8 to 1.2, with an increment of 0.05. Suppose 1 year has 252 trading days, then the sample interval is $1/252$.

Figure 2 depicts the plots of option prices against spot-to-strike ratio for each of the fixed maturities $T = 0.5, 1.0$ and $1.5$. From Figure 2, we can find that the jump

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Value in state $e_1$</th>
<th>Value in state $e_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual domestic interest rate</td>
<td>$r^D_{1} = 0.04$</td>
<td>$r^D_{2} = 0.01$</td>
</tr>
<tr>
<td>Annual foreign interest rate</td>
<td>$r^F_{1} = 0.025$</td>
<td>$r^F_{2} = 0.03$</td>
</tr>
<tr>
<td>Appreciation rate of S</td>
<td>$\mu_1 = 0.05$</td>
<td>$\mu_2 = 0.02$</td>
</tr>
<tr>
<td>Volatility in model (2.2)</td>
<td>$\sigma_1 = 0.1$</td>
<td>$\sigma_2 = 0.3$</td>
</tr>
<tr>
<td>Annual jump intensity</td>
<td>$\lambda_1 = 5$</td>
<td>$\lambda_2 = 10$</td>
</tr>
</tbody>
</table>
Table 2. Parameters in two-factor Markov-modulated models.

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial variance $V_0$</td>
<td>$V_0 = 0.01$</td>
</tr>
<tr>
<td>Volatility of $V$</td>
<td>$\sigma = 0.1$</td>
</tr>
<tr>
<td>Correlation coefficient $\rho$</td>
<td>$\rho = 0.5$</td>
</tr>
<tr>
<td>Initial FX rate $S_0$</td>
<td>$S_0 = 1$</td>
</tr>
<tr>
<td>Initial state of Markov chain $\xi$</td>
<td>1</td>
</tr>
<tr>
<td>Mean jump size $\mu$</td>
<td>$\mu = 0$</td>
</tr>
<tr>
<td>Standard deviation of the jump size $\sigma_j$</td>
<td>$\sigma_j = 0.1$</td>
</tr>
<tr>
<td>Speed of reversion of $V$ $\beta$</td>
<td>$\beta = 2$</td>
</tr>
<tr>
<td>Long-term average of $V$ $\gamma/\beta$</td>
<td>$\gamma/\beta = 0.01$</td>
</tr>
</tbody>
</table>

Figure 2. Option price against spot-to-strike ratio. The solid, dashed and dotted lines correspond to models (2.1), Bo et al. (2010) and Siu et al. (2008), respectively.

Risk in our model seems to have a more significant impact on option prices than that of stochastic volatility. Compared with Bo et al. (2010), our model considers the long-term fluctuation. Due to the additional risk in our consideration, the values in the
The proposed model should be higher than those of the two reference models. Although the option price turns higher as the spot-to-strike ratio increases, the jump risk and the long-term fluctuation may be independent of the strike prices since the lines are almost parallel. Furthermore, we find that in Figure 2 the option prices in Bo et al. (2010) are higher than those of Siu et al. (2008). In their models, the jump risk has a more significant impact on the price than the long-term risk does.

Figure 3 depicts the plots of the option prices against the maturities for each of the fixed spot-to-strike ratios $S_0/K = 0.8$, $1.0$ and $1.2$. The option prices calculated by two-factor Markov-modulated jump-diffusions are higher than those of Bo et al. (2010) and Siu et al. (2008), which agrees with the observation from Figure 2. Moreover, since the distance between the dotted line and the solid line becomes farther as the maturity increases, the jump risk seems more pronounced. The result is also true for

\[ S_0/K = 0.8 \]

\[ S_0/K = 1.0 \]

\[ S_0/K = 1.2 \]

**Figure 3.** Option price against time to maturity. The dotted, dashed and solid lines correspond to models (2.1), Bo et al. (2010) and Siu et al. (2008), respectively.
the long-term fluctuation. However, this phenomenon is not obvious when comparing the model of Bo et al. (2010) with that of Siu et al. (2008). We find that the distance between the dashed line and the solid line becomes farther at the beginning, then a little nearer to each other. In our adopted parameters, the jump intensity has a more significant impact on the option price than long-term volatility does. But when domestic macroeconomic shifts to state $e_2$, long-term volatility changes more than the jump intensity, which may shorten the distance between the dashed and the solid lines.

Finally, we investigate how the long-term volatility $\sigma_t$ and the annual jump intensity $\lambda_t$ affect option prices. Figure 4 presents the prices varying with the annual jump intensity and Figure 5 depicts the counterpart as the long-term volatility changes. We change the parameters by the same percentage while the distances among the lines in Figure 4 are farther than those in Figure 5, which can be inferred that the effect of the annual jump intensity is more remarkable than that of the long-term volatility. Similar to Bo et al. (2010), asymmetric phenomenon exists in our model.

6. Conclusion

In this paper, the spot FX rate is assumed to be driven by a two-factor Markov-modulated jump-diffusion process. Compared with most of the existing Markov-modulated diffusion models, the main advantage of the proposed model is that short-term risk, long-term risk and rare shocks are considered together.
Random Esscher transform is adopted to determine an equivalent martingale measure. We decompose the log FX rate into a continuous part and a jump part, and then get the closed-form solutions for the regime-switching Esscher parameters (risk premiums). Under the equivalent domestic martingale measure, we give the pricing formula of European-style currency options.

In the section of numerical illustrations, we compare our model with those of Bo et al. (2010) and Siu et al. (2008). We find that the risk from rare events has a more significant impact on the option prices than the continuous Brownian part does. It is reasonable to combine short-term risk, long-term risk and jump risk when pricing currency options.

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References

Appendix

Proof of Proposition 3.1. Recall that \( \mathcal{G}_t = \mathcal{F}_t^\xi \vee \mathcal{F}_t^Y \). Thanks to the mutual independence of \( W^1, W^2, N \) and \( Z = (Z_t)_{0 \leq t \leq T} \), we can show that

\[
E \left[ \exp \left( \int_0^t \theta_s^e dC_s + \int_0^t \theta_s^J dJ_s \right) \mid \mathcal{G}_t \right]
= E \left[ \exp \left( \int_0^t \theta_s^e (\alpha_s - k\lambda_s - \frac{1}{2} (V_s + \sigma_s^2)) ds + \int_0^t \theta_s^J \sqrt{V_s} dW_s^1 + \int_0^t \theta_s^J \sigma_s dW_s^2 + \int_0^t \theta_s^J Z_s dN_s \right) \mid \mathcal{G}_t \right]
\times E \left[ \exp \left( \int_0^t \theta_s^J Z_s dN_s \right) \mid \mathcal{G}_t \right]
= \exp \left( \int_0^t \theta_s^e (\alpha_s - k\lambda_s - \frac{1}{2} (V_s + \sigma_s^2)) ds + \frac{1}{2} \int_0^t (V_s + \sigma_s^2)(\theta_s^e)^2 ds \right)
\times \exp \left( \int_0^t \lambda_s \left( e^{\theta_s^J \mu_s + \frac{1}{2} (\theta_s^J \sigma_s)^2} - 1 \right) ds \right). \tag{A.1}
\]

Substituting Equation (A.1) into Equation (3.2), the following equations hold:

\[
P_{\alpha, \beta}^t = \frac{\exp \left( \int_0^t \theta_s^e (\alpha_s - k\lambda_s - \frac{1}{2} (V_s + \sigma_s^2)) ds + \int_0^t \theta_s^e \sqrt{V_s} dW_s^1 + \int_0^t \theta_s^e \sigma_s dW_s^2 + \int_0^t \theta_s^e Z_s dN_s \right)}{\exp \left( \int_0^t \theta_s^e (\alpha_s - k\lambda_s - \frac{1}{2} (V_s + \sigma_s^2)) ds + \frac{1}{2} \int_0^t (V_s + \sigma_s^2)(\theta_s^e)^2 ds + \int_0^t \lambda_s \left( e^{\theta_s^J \mu_s + \frac{1}{2} (\theta_s^J \sigma_s)^2} - 1 \right) ds \right)}
= \exp \left( \int_0^t \theta_s^e \sqrt{V_s} dW_s^1 + \int_0^t \theta_s^e \sigma_s dW_s^2 - \frac{1}{2} \int_0^t (V_s + \sigma_s^2)(\theta_s^e)^2 ds \right)
\times \exp \left( \int_0^t \theta_s^e Z_s dN_s - \int_0^t \lambda_s \left( e^{\theta_s^J \mu_s + \frac{1}{2} (\theta_s^J \sigma_s)^2} - 1 \right) ds \right).
\]

Thus, the proof is completed.
Proof of Lemma 3.1. Denote $\mathbb{E}^{\theta, \beta'}$ the mathematical expectation operator with respect to the Esscher transform $Q^{\theta, \beta'} \sim P$. By the Bayes' formula, for $u < t$, one gets that

$$\mathbb{E}^{\theta, \beta'}[S_t^d|\mathcal{H}_u] = \mathbb{E}\left[\frac{L_t^{\theta, \beta'} S_t^d|\mathcal{H}_u}{L_u^{\theta, \beta'}}\right] = \mathbb{E}\left[\frac{L_t^{\theta, \beta'} S_t^d|\mathcal{H}_u}{L_u^{\theta, \beta'}}\right]. \tag{A.2}$$

On the other hand, Itô rules for general semi-martingales imply that

$$S_t^d = S_u^d \exp\left(\int_u^t (r_s^F - r_s^D + \alpha_s - k\lambda_s - \frac{1}{2}(\sigma_s^2 + V_s))ds\right) \times \exp\left(\int_u^t \sqrt{V_s}dW_s^1 + \int_u^t \sigma_s dW_s^2 + \int_u^t Z_s dN_s\right).$$

Then, the following equations hold:

$$\frac{L_t^{\theta, \beta'} S_t^d}{L_u^{\theta, \beta'} S_u^d} = \exp\left(\int_u^t (r_s^F - r_s^D + \alpha_s - k\lambda_s + \theta_s^c(\sigma_s^2 + V_s))ds + M_{u,t}^c\right) \times \exp\left(\int_u^t \lambda_s \left(e^{(1+\theta_s^c)\mu_J + \frac{1}{2}(1+\theta_s^c)^2\sigma_J^2} - 1\right) ds\right) \times \exp\left(-\int_u^t \lambda_s \left(e^{\theta_s^c \mu_J + \frac{1}{2}(\theta_s^c)^2\sigma_J^2} - 1\right) ds + M_{u,t}^J\right), \tag{A.3}$$

where for $u < t \leq T$,

$$M_{u,t}^c = \int_u^t (1 + \theta_s^c)\sqrt{V_s}dW_s^1 + \int_u^t (1 + \theta_s^c)\sigma_s dW_s^2 - \frac{1}{2} \int_u^t (1 + \theta_s^c)^2(\sigma_s^2 + V_s)ds,$$

$$M_{u,t}^J = \int_u^t (1 + \theta_{s-}^J)Z_{s-}dN_s - \int_u^t \lambda_s \left(e^{(1+\theta_s^J)\mu_J + \frac{1}{2}(1+\theta_s^J)^2\sigma_J^2} - 1\right) ds.$$

Based on the assumption that $\mathbb{E}e^{\frac{1}{2} \int_0^T (1+\theta_s^J)^2 V_s ds} < \infty$, it is clear that $M_{u,t}^c$ is an exponential martingale. Hence, combining Equations (A.2) and (A.3), it follows that
Finally, the martingale condition $E^{\theta^c, \theta^j} S_t^d | \mathcal{H}_u = S_u^d$ holds if and only if the Markov-modulated parameters $(\theta^c, \theta^j)$ satisfy

$$r_t^F - r_t^D + \alpha_t - k \lambda_t + \theta^c_s(\sigma_s^2 + V_s) + \lambda^0 \theta^j t = 0,$$

for all $0 \leq t \leq T$.

**Proof of Proposition 3.2.** Obviously, $(\theta_t^{c, *}, \theta_t^{j, *})_{0 \leq t \leq T}$ satisfy the martingale condition (3.3). In the following, we show that $(\theta_t^{c, *}, \theta_t^{j, *})_{0 \leq t \leq T}$ satisfy both $E e^{\frac{1}{2} \int_0^T \theta_t^{c, *} \sigma_t^2 V_t \, dt} < \infty$ and $E e^{\frac{1}{2} \int_0^T (1 + \theta_t^{c, *})^2 V_t \, dt} < \infty$. Recall that $\theta_t^{c, *} = \frac{k \lambda_t - \alpha_t + r_t^D - r_t^F}{V_t + \sigma_t^2}$ in Equation (3.4), it holds that

$$\left( \frac{k \lambda_t - \alpha_t + r_t^D - r_t^F}{V_t + \sigma_t^2} \right)^2 V_t = \left( \frac{k \lambda_t - \alpha_t + r_t^D - r_t^F}{\sqrt{V_t} + \frac{\sigma_t^2}{\sqrt{V_t}}} \right)^2 \leq \frac{(k \lambda_t - \alpha_t + r_t^D - r_t^F)^2}{4 \sigma_t^2}.$$

Therefore, it is clear that

$$E e^{\frac{1}{2} \int_0^T \theta_t^{c, *} \sigma_t^2 V_t \, dt} \leq E e^{\frac{1}{2} \int_0^T \frac{(k \lambda_t - \alpha_t + r_t^D - r_t^F)^2}{4 \sigma_t^2} \, dt} < \infty.$$

On the other hand, we have that

$$E e^{\frac{1}{2} \int_0^T (1 + \theta_t^{c, *})^2 V_t \, dt} \leq E e^{\frac{1}{2} \int_0^T (1 + \theta_t^{c, *} \sigma_t^2) V_t \, dt}$$

$$= E e^{\frac{1}{2} \int_0^T V_t \, dt + \int_0^T \theta_t^{c, *} \sigma_t^2 V_t \, dt}$$

$$\leq E \left( e^{\frac{1}{2} \int_0^T V_t \, dt} + e^{\frac{1}{2} \int_0^T \theta_t^{c, *} \sigma_t^2 V_t \, dt} \right)$$

$$= \frac{1}{2} E e^{\frac{3}{2} \int_0^T V_t \, dt} + \frac{1}{2} E e^{\frac{1}{2} \int_0^T \theta_t^{c, *} \sigma_t^2 V_t \, dt}.$$
Similarly, one gets that

\[ \frac{1}{2} \mathbb{E} e^{\int_0^T (\theta^c_t)^2 V_t \, dt} < \infty. \]

Note that \( V_t \) satisfies the following stochastic differential equation:

\[ dV_t = (\gamma - \beta V_t) \, dt + \sigma \sqrt{V_t} \, dW_t. \]

Actually, we can write it as follows:

\[ V_T = V_0 + \gamma T - \beta \int_0^T V_t \, dt + \sigma \int_0^T \sqrt{V_t} \, dW_t. \]

Since \( V_t \geq 0 \), for \( t \geq 0 \), we have that

\[ \int_0^T V_t \, dt = \frac{1}{\beta} (V_0 - V_T + \gamma T + \sigma \int_0^T \sqrt{V_t} \, dW_t) \]

\[ \leq \frac{1}{\beta} (V_0 + \gamma T + \sigma \int_0^T \sqrt{V_t} \, dW_t). \]

Taking expectation on both sides,

\[ \mathbb{E} \int_0^T V_t \, dt \leq \frac{1}{\beta} (V_0 + \gamma T) < \infty, \]

which guarantees \( \int_0^T V_t \, dt \) and \( \int_0^T \sqrt{V_t} \, dW_t \) are well defined.

From the fact that \( \int_0^T \sqrt{V_t} \, dW_t \) is normally distributed, we can imply that

\[ \mathbb{E} e^{\int_0^T V_t \, dt} < \infty. \]

Thus, the proof is completed.

**Proof of Lemma 3.2.** We follow the method in Bo et al. (2010). Since \( W_1 \), \( W_2 \) and \( N \) are mutually independent, Girsanov theorem helps us to derive that \( \tilde{W}_t^{1,\theta^c} \) defined in Equation (3.8) and \( \tilde{W}_t^{2,\theta^c} \) defined in Equation (3.9) are Brownian motions under \( Q^{\theta^c,\theta^h} \). Next we show Equations (3.10) and (3.11) hold conditional on \( G_T \). Let \( dM_t = dN_t - \lambda \, dt \) be a \( P \)-martingale. Then, Girsanov theorem for point processes (see, e.g., Protter, 1990) implies that

\[ dM_t^{\theta^c,\theta^h} = dM_t - \lambda_t \left( e^{\theta_t^c \mu_j + \frac{1}{2} (\theta_t^c \sigma_j)^2} - 1 \right) \, dt \]

\[ = dN_t - \lambda_t e^{\theta_t^c \mu_j + \frac{1}{2} (\theta_t^c \sigma_j)^2} \, dt \]

is a \( Q^{\theta^c,\theta^h} \)-martingale. Thus, Equation (3.10) holds.
Similarly, we could define a $\mathbb{P}$-martingale $(M_t^Z)_{0 \leq t \leq T}$ as
\[
dM_t^Z = (e^{Z_t} - 1)S_t dN_t - k\lambda_t S_t d\tau.
\]
By Girsanov theorem, the following equations hold:
\[
dM_t^{Z, \theta^J, \theta^J} = dM_t^Z - \lambda_t E \left[ \left( e^{\theta_t^J Z_t} - 1 \right) \left( e^{Z_t} - 1 \right) | \mathcal{F}_t^\xi \right] S_t d\tau
\]
\[
= (e^{Z_t} - 1)S_t dN_t - \lambda_t^{\theta_t^J} k_t^{\theta_t^J} S_t d\tau,
\]
which is a $Q^{\theta^J, \theta^J}$-martingale. Then Equation (3.11) holds.